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Note

The new FIFA rules are hard: complexity aspects of sports competitions

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Abstract

Consider a soccer competition among various teams playing against each other in pairs (matches) according to a previously determined schedule. At some stage of the competition one may ask whether a particular team still has a (theoretical) chance to win the competition. The complexity of this question depends on the way scores are allocated according to the outcome of a match. For example, the problem is polynomially solvable for the ancient FIFA rules (2:0 resp. 1:1) but becomes NP-hard if the new rules (3:0 resp. 1:1) are applied. We determine the complexity of the above problem for all possible score allocation rules. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Consider a sports competition like a national soccer league in which all participating teams play against each other in pairs (matches) according to a prefixed schedule. Initially, all teams have total score zero. When a team participates in a match, its total score is increased by $\alpha \in \mathbb{R}$ if it loses the match, by $\beta \in \mathbb{R}$ if the match ends in a draw, and by $\gamma \in \mathbb{R}$ if it wins the match. We always assume that $\alpha \leqslant \beta \leqslant \gamma$ and call the triple (α, β, γ) the *rule* (score allocation rule) of the competition. In case of a soccer competition, the former FIFA rule was $(\alpha, \beta, \gamma) = (0, 1, 2)$, but this has been changed into the new rule $(\alpha, \beta, \gamma) = (0, 1, 3)$. Other sports like chess or draughts still use the rule $(\alpha, \beta, \gamma) = (0, 1, 2)$, while stratego, also a strategic board game, has as score allocation rule $(\alpha, \beta, \gamma) = (0, 1, 6)$.

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At a given stage of the competition one may ask whether a particular team T_0 still has a (theoretical) chance of "winning" the competition, i.e., ending up with the highest final total score. To analyze this, we may w.l.o.g. assume that T_0 wins all remaining matches, resulting in a final total score s_0 for T_0 and a current total score s_i for all other teams $T_i \neq T_0$. The question is now whether the teams $T_i \neq T_0$ can finish the remaining matches in such a way that each T_i collects at most $c_i := s_0 - s_i$ additional score points.

This can be modeled by a multigraph $G = (V, E)$ whose vertices correspond to teams $T_i \neq T_0$ and edges are in 1–1 correspondence with remaining matches. Each node $i \in V$ has a capacity $c_i \in \mathbb{R}$. We represent the outcome of a match $e = (i, j)$ by directing the edge from the winner to the loser (and leaving the edge undirected in case of a draw). Our sports competition problem ("SC") can now be formulated as follows:

SC(α, β, γ)

Given a multigraph $G = (V, E)$ and node capacities $c \in \mathbb{R}^V$ can G be partially oriented such that for each node $i \in V$:

$$\alpha\delta^-(i) + \beta\delta^0(i) + \gamma\delta^+(i) \leqslant c_i? \tag{1.1}$$

Here, as usual, δ^+ and δ^- denote the outdegree and indegree of a node, whereas δ^0 denotes the number of incident unoriented edges. A partial orientation of G satisfying capacity constraints (1.1) is called a *solution* of the instance (G, c).

A simplified version of this (disallowing draws) was presented in Cook et al. [1]. In this case, the problem reduces to a flow problem, cf. Cook et al. [1] or Section 2 below. As we shall see, however, the question becomes more interesting if draws may occur. Our main result implies that in this case the problem is polynomially solvable if $\alpha + \gamma = 2\beta$ (assuming P $\neq$ NP). This means that for games like draughts and chess the problem is polynomially solvable. However, for soccer competitions, by changing the score allocation rule into the rule $(\alpha, \beta, \gamma) = (0, 1, 3)$, the problem has become NP-complete. Also for stratego competitions the problem is NP-complete.

We end our introduction with the following simple observation. Given an instance (G, c) of SC(α, β, γ), we can derive an equivalent instance (G, c') of SC$(0, \beta - \alpha, \gamma - \alpha)$ by setting $c_i' := c_i - \alpha\delta(i)$. (Here, δ refers to the degree in G.) So with respect to computational complexity of SC(α, β, γ) we may always assume that (α, β, γ) is *normalized*, i.e., $\alpha = 0 \leqslant \beta \leqslant \gamma$.

2. Complexity results

Our main result completely determines the computational complexity of the sports competition problem.

Theorem 2.1. *SC(α, β, γ) is polynomially solvable in each of the following three cases*:
(i) $\alpha = \beta$,

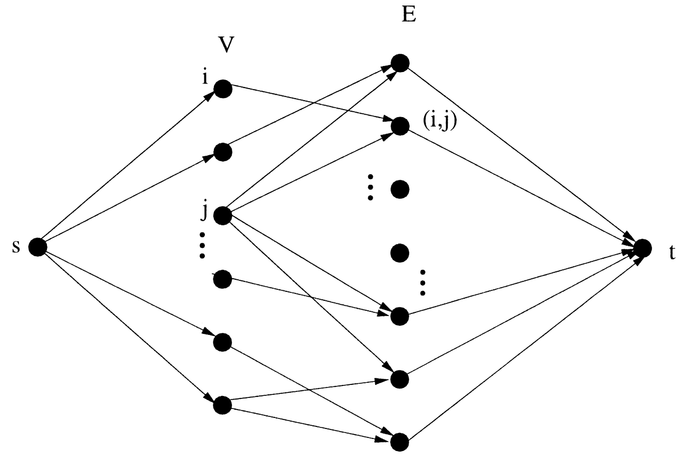

Fig. 1.

(ii) $\beta = \gamma$,
(iii) $\alpha + \gamma = 2\beta$.
In all other cases, the problem is NP-*complete*.

Proof. First recall that we may assume (α, β, γ) is normalized, so $\alpha = 0$. (Note that normalization does not affect (i)–(iii).) **Case (i)** is then trivial. Indeed, an instance (G, c) has a solution if and only if $c \geqslant 0$. (Leave all edges unoriented.)

In all other cases we have $\beta > 0$. By scaling, we may assume that $\beta = 1$. (Divide β, γ as well as c by β.)

Case (ii) $\beta = \gamma = 1$.

Consider an instance given by $G = (V, E)$ and $c \in \mathbb{R}^V$. Construct a directed bipartite graph with node sets V and E and arcs linking each $i \in V$ to all edges in E incident with i in G. Then add an additional source s and sink t as indicated in Fig. 1.

The arcs from s to V all get lower capacity 0 and upper capacity $\lfloor c_i \rfloor$ $(i \in V)$. The arcs from V to E get lower capacity 0 and upper capacity 1. The arcs from E to t get lower and upper capacity 1. The resulting network has a feasible $s - t$ flow $x \in \mathbb{R}^{|V|+3|E|}$ if and only if our instance (G, c) has a solution. Indeed, as all capacities are integral, a feasible flow may also be assumed to be integral. Given an integral feasible flow we can interpret an arc $(i, (i, j))$ from V to E which carries 1 unit of flow as i winning the match $e = (i, j)$ and conversely (cf. also [1]).

Case (iii) $\beta = 1$, $\gamma = 2$ (ancient FIFA rule).

This can be solved similarly. In the network of Fig. 1 we simply redefine the upper capacities of all arcs from V to E to be 2. The lower and upper capacities of arcs from E to t are also set to 2. Again, feasible integral flows are in 1–1 correspondence with solutions of our instance (G, c). Each node $e \in E$ in our network has two incoming arcs which carry a total flow of 2 units, distributed as 2:0 or 1:1, corresponding to a win/loss match or a draw.

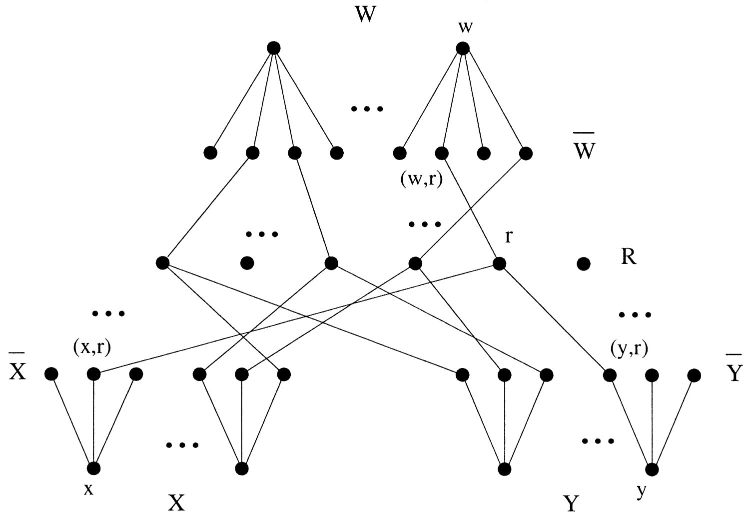

Fig. 2.

Case (iv) $\beta = 1,\ \gamma > 2$.

We prove NP-completeness by reduction from three-dimensional matching (cf. [2]). Suppose $|X| = |Y| = |W| = q$ and $R \subseteq X \times Y \times W$ is given. We are to determine whether R contains a matching $R' \subseteq R$, i.e., a set of triples covering each element of $X \cup Y \cup W$ exactly once. Assume w.l.o.g. that each element $z \in X \cup Y \cup W$ actually occurs in some triple $r \in R$. We write $z \in r$ to indicate that z occurs in $r \in R$. Given $R \subseteq X \times Y \times W$, we construct a graph $G = (V, E)$ as follows. We first make one copy of each element $z \in X \cup Y \cup W$ for each occurrence of z in R, i.e., we define

$$\bar{X} := \{(x, r) \mid x \in X, r \in R, x \in r\},$$

$$\bar{Y} := \{(y, r) \mid y \in Y, r \in R, y \in r\},$$

$$\bar{W} := \{(w, r) \mid w \in W, r \in R, w \in r\}.$$

Construct a graph $G = (V, E)$ with node set $V = X \cup Y \cup W \cup \bar{X} \cup \bar{Y} \cup \bar{W} \cup R$ and edges as defined by the incidence relations in a straightforward way, i.e.,

$$\begin{aligned} E = &\{(x, (x, r)) \mid (x, r) \in \bar{X}\} \\ &\cup \{(y, (y, r)) \mid (y, r) \in \bar{Y}\} \\ &\cup \{(w, (w, r)) \mid (w, r) \in \bar{W}\} \\ &\cup \{(r, (x, r)) \mid (x, r) \in \bar{X}\} \\ &\cup \{(r, (y, r)) \mid (y, r) \in \bar{Y}\} \\ &\cup \{(r, (w, r)) \mid (w, r) \in \bar{W}\} \quad \text{(cf. Fig. 2).} \end{aligned}$$

Next, define node capacities $c \in \mathbb{R}^V$ as follows:

$c \equiv 1$ on $X \cup Y$,
$c \equiv 1 + \gamma$ on $\bar{X} \cup \bar{Y}$,

$$
\begin{aligned}
&c \equiv \max\{\gamma, 3\} && \text{on } R,\\
&c \equiv 1 && \text{on } \bar{W},\\
&c \equiv \gamma(\delta - 1) + 1 && \text{on } W.
\end{aligned}
$$

(Again, δ refers to the degree function of G.)

We claim that this instance (G, c) has a solution if and only if R contains a matching.

"$\Leftarrow$" Suppose $R' \subseteq R$ is a matching. Define a corresponding partial orientation of G as follows. For each $w \in W$ choose the unique $r' \in R'$ with $(w, r') \in \bar{W}$. We leave the edge $(w, (w, r'))$ unoriented and orient all other edges from w to $\bar{W}$. This way the capacity constraints of w are met. For each $r' = (x, y, w) \in R'$ we orient the edge $(r', (w, r'))$ from r' towards (w, r') and the edges $(r', (x, r'))$ and $(r', (y, r'))$ from $\bar{X}$, respectively, $\bar{Y}$ towards r'. All edges incident with $r \in R \setminus R'$ remain unoriented. This way we ensure that the capacity constraints on $\bar{W}$ and R are respected. Finally, orient all edges between $\bar{X}$ and X from $\bar{X}$ towards X except those that correspond to an element in R' (these remain unoriented). This way the capacity constraints for X and $\bar{X}$ are met. We orient edges between $\bar{Y}$ and Y in the same way. This partial orientation gives a solution of the instance (G, c).

"$\Rightarrow$" Conversely, suppose we are given a partial orientation of G respecting the capacity constraints. The latter imply that for $x \in X$ we have $\delta^-(x) \geqslant \delta(x) - 1$ and $\delta^+(x) = 0$. We may assume w.l.o.g. that actually $\delta^-(x) = \delta(x) - 1$. (Otherwise, i.e., if $\delta^-(x) = \delta(x)$, pick an arbitrary edge incident with x and make it unoriented. The modified orientation will still respect all capacity constraints.) A similar argument holds for elements $y \in Y$. Nodes in $\bar{X}$ have degree 2. In view of their capacity bound $1 + \gamma$, we may assume w.l.o.g. that each $(x, r) \in \bar{X}$ has $\delta^0 = 1$ and $\delta^+ = 1$. (Otherwise, again modify the solution without violating the capacity constraints.) As each $x \in X$ has $\delta^-(x) = \delta(x) - 1$ and $\delta^0(x) = 1$, we conclude that

- There are exactly $|X|$ arcs directed from $\bar{X}$ to R. Moreover, if $((x, r), r)$ is directed towards r and $((x', r'), r')$ is directed towards r', then $x \neq x'$.

The same holds for the directed arcs from $\bar{Y}$ to R.

Arguing similarly for nodes in W, we find that each $w \in W$ has w.l.o.g. $\delta^+(w) = \delta(w) - 1$ and $\delta^0(w) = 1$. (Otherwise modify the orientation such that w actually uses its full capacity.) Because nodes in $\bar{W}$ have degree 2 and capacity bound 1, this implies that

- There are exactly $|W|$ arcs directed from R towards $\bar{W}$. Moreover, if $(r, (w, r))$ is directed from r towards (w, r) and $(r', (w', r'))$ is directed from r' towards (w', r'), then $w \neq w'$.

Finally, the capacity constraints on R imply that a node $r \in R$ can have $\delta^+ \geqslant 1$ only if $\delta^- \geqslant 2$. From this and the above observations, it is straightforward to check that

$$R' = \{r \in R \mid \delta^+(r) = 1\}$$

actually is matching.

Case (v) $\beta = 1 < \gamma < 2$.

Again, we prove NP-completeness by reduction from three-dimensional matching. In the graph G of Fig. 2 we redefine the node capacities $c \in \mathbb{R}^V$ as follows:

$$\begin{aligned} c &\equiv \gamma(\delta - 1) + 1 && \text{on } X \cup Y, \\ c &\equiv 1 && \text{on } \bar{X} \cup \bar{Y}, \\ c &\equiv \max\{2\gamma, 3\} && \text{on } R, \\ c &\equiv 1 + \gamma && \text{on } \bar{W}, \\ c &\equiv 1 && \text{on } W. \end{aligned}$$

Analogously to Case (iv) one can prove that the instance (G, c) has a solution if and only if R contains a matching. □

3. Remarks

As noted already, our results imply that sport competition problems with the new FIFA rules ($\alpha = 0$, $\beta = 1$, $\gamma = 3$) are hard. The reason for this is that the network model we used for solving cases (ii) and (iii) of our main theorem does not apply for this case. Indeed, if we increase the upper capacities to 3 on all arcs from V to E and from E to t in the network of Fig. 1, then a feasible flow does no longer necessarily represent a solution of our instance. (A total flow of 2 entering a node $e = (i, j) \in E$ distributed as 2:0 on the two entering arcs does not correspond to a win/loss or a draw.) If we "repair" this by introducing a "capacity gap" $]1, 3[$ on all arcs from V to E we get a flow problem with capacity gaps which again nicely describes our sports competition problem. So as a consequence of our result, the following class of problems is also NP-complete (this might be known, but we could not find it in the literature):

Flows with capacity gaps ("FCG")

Instance: A digraph $D = (V, A)$ with source s and sink t and for each arc $a \in A$ two disjoint capacity intervals $I_1(a) = [c_1(a), c_2(a)]$ and $I_2(a) = [c_3(a), c_4(a)]$ ($c_i(a) \in \mathbb{Z}$, $i = 1, \ldots, 4$).

Question: Does a (w.l.o.g. integral) $s - t$ flow $x \in \mathbb{Z}^A$ exist with $x(a) \in I_1(a) \cup I_2(a)$ $(a \in A)$?

Corollary 3.1. *FCG is* NP-*complete*.

Finally, as to sports competitions, we would like to remark that also other questions can be treated in the same way. For example "Is there a chance that T_0 ends up with the lowest final score?" turns out to be of exactly the same complexity as SC: Assume that T_0 has a current total score s_0 and loses all remaining matches. This results in a current total score s_i for all other teams $T_i \neq T_0$. The first question is now whether the teams $T_i \neq T_0$ can finish the remaining matches in such a way that each T_i collects *at least* $c_i := s_0 - s_i$ additional score points. Again, we model this by a multigraph $G = (V, E)$ whose vertices correspond to teams $T_i \neq T_0$ and edges are in 1–1 correspondence with remaining matches. Each node $i \in V$ has a (lower) capacity $c_i \in \mathbb{R}$. Our "reverse" sports competition problem ("RSC") can now be formulated as follows:

RSC(α, β, γ):

Given a multigraph $G=(V,E)$ and node capacities $c \in \mathbb{R}^V$ can G be partially oriented such that for each node $i \in V$:

$$\alpha\delta^-(i) + \beta\delta^0(i) + \gamma\delta^+(i) \geqslant c_i? \tag{3.1}$$

It is easy to see that for $i \in V$, (3.1) is equivalent to

$$(\gamma - \beta)\delta^0(i) + (\gamma - \alpha)\delta^-(i) \leqslant \gamma\delta(i) - c_i.$$

Hence an instance (G, c) of RSC(α, β, γ) corresponds to an instance $(G, \gamma\delta - c)$ of SC($0, \gamma - \beta, \gamma - \alpha$) and the corollary below immediately follows from Theorem 2.1.

Corollary 3.2. *RSC(α, β, γ) is polynomially solvable in each of the following three cases*:
(i) $\alpha = \beta$,
(ii) $\beta = \gamma$,
(iii) $\alpha + \gamma = 2\beta$.
In all other cases, *the problem is* NP-*complete*.

Questions such as "Is there a chance that T_0 ends up being one of the three teams that have the three lowest final scores?" can also be treated in a similar way. Again, assume that T_0 has a current total score s_0 and loses all remaining matches. Furthermore, choose two teams $T_i, T_j \neq T_0$ and let T_i and T_j lose their remaining matches against teams T_k ($k \neq 0, i, j$). (Choose, if necessary, an arbitrary outcome for the matches between T_i and T_j.) These outcomes result in final total scores s_0, s_i and s_j, and current total scores s_k for all other teams T_k ($k \neq 0, i, j$). If it is possible that the teams T_k ($k \neq 0, i, j$) can finish the remaining matches in such a way that each T_k collects at least $c_k := s_0 - s_k$ additional score points, then T_0 can indeed end up being one of the three lowest teams. If this is not possible for any pair T_i, T_j, then T_0 can never end up being one of the three lowest teams. So one has to solve at most $\frac{1}{2}|V|(|V| - 1)$ problem instances in RSC(α, β, γ). Hence also this question is of the same complexity as SC.

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